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Numerical solutions of generalized Abel integral and integro differential equations

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Abstract

This paper introduced a numerical method for solving generalized Abel integral (GAI) equations and generalized Abel integro differential (GAID) equations. This method is based upon Touchard polynomials (TPs) approximation. The Touchard polynomials were first presented and the resulting Touchard matrices were utilized to transform the generalized Abel integral and integro differential equations into a system of linear algebraic equations. The results of the presented method were obtained through some examples of the first and second types of equations under study. All results for this method have been compared with those of the presented methods in the literature.

Subject Classification: *20C10; 22E27.*

Keywords: *Generalized Abel integral equation, Generalized Abel integro differential equation, Touchard polynomials, Numerical solution, Exact solution.*

1. Introduction

The following is a summary of the Abel problem: Niles Abel is a Norwegian mathematician who published a paper in 1823 aiming to

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study a smooth vertical curve on the *x-y* plane that allows a particle to fall without friction from the highest known point to the origin point using only gravity and no initial velocity [1, 2, 3]. In other words, the Integral equations ofAbel have a wide range of applications: Mathematical physics, semiconductors, chemistry, heat conduction, electrochemistry, scattering theory, chemical reactions, seismology, metallurgy, fluid movement, and population dynamics etc. are just a few examples [1, 2,3,4]. Many scientists, mathematicians, and physicists have worked hard in recent years to find numerical solutions for generalized Abel integral and integro differential equations, among these strategies: Laplace transform by power series [4], Taylor collocation method [5], the barycentric rational interpolation technique and the Legendre Gauss quadrature method [6], homotopy perturbation technique [7], Bernstein polynomials method and Legendre wavelets [8], Lagrangian basis functions technique [9], finally Chebyshev polynomials method [10]. This paper's structure is as follows: Method of the solution, Approximation function, solutions generalized Abel integral equations, solution accuracy, test with the illustrative examples, plot the graphs are presented, brief of conclusions, the references are also listed.

The generalized Abel integral equations of the second and first types are [4, 5] respectively have the forms:

$$N(u) = m(u) + \int_{\rho}^{u} \frac{1}{(u-\mu)^{\tau}} N(\mu)\, d\mu, \quad 0 < \tau < 1 \qquad \ldots(1)$$

$$m(u) = \int_{\rho}^{u} \frac{1}{(u-\mu)^{\tau}} N(\mu)\, d\mu, \quad 0 < \tau < 1 \qquad \ldots(2)$$

and the 2nd type of generalized Abel integro differential equation [8, 9, 10] has the form:

$$N'(u) = m(u) + \int_{\rho}^{u} \frac{1}{(u-\mu)^{\tau}} N(\mu)\, d\mu, \quad 0 < \tau < 1 \qquad \ldots(3)$$

with initial condition $N(0) = N_0$, where $N'(u) = \frac{dN(u)}{du}$, ρ is a given real number value, $m(u)$ is a well-known function, while $N(u)$ is unknown.

2. Method of the Solution

Let's start [11, 12, 13, 14, 15] with the definition and description of Touchard polynomials, which are polynomial sequences the binomial type specified over [0, 1], as developed by French mathematician Jacques

Touchard (1885-1968). His name was given to these polynomials, which represent the following formula:

$$y_n(u)=\sum_{a=0}^{n}\sigma\ (n,a)u^a=\sum_{a=0}^{n}\binom{n}{a}u^a\ ,\quad \binom{n}{a}=\frac{n!}{a!(n-a)!}\ , \qquad \ldots(4)$$

where, a and n denote to the index and degree respectively of Touchard polynomials. The first six of these polynomials are as follows:

2.1 *Approximation Function*

Assume that the function $N_n(u)$ is approximated as follows using the (TPs):

$$N_n(u)=p_0y_0(u)+p_1y_1(u)+\ldots+p_ny_n(u)=\sum_{a=0}^{n}p_ay_a(u), \qquad \ldots(5)$$

where $0 \le u \le 1$, for $a \ge 0$, the function $\{y_a(u)\}_{a=0}^{n}$ denotes *nth*-degree of Touchard basis polynomials, as defined in equation (4), p_a*'s* are unknown parameters that will be provided later and n any positive integer number. Equation (5) can now be expressed as a dot product:

$$N_n(u)=[y_0(u)\,y_1(u)\ldots y_n(u)]\ .\begin{bmatrix}p_0\\p_1\\.\\.\\.\\p_n\end{bmatrix}, \qquad \cdots(6)$$

equation (6) can be written as:

$$N_n(u)=[1\ u\ u^2\ldots u^n].\begin{bmatrix}\vartheta_{00} & \vartheta_{01} & \vartheta_{02} & \cdots & \vartheta_{0n}\\ 0 & \vartheta_{11} & \vartheta_{12} & \cdots & \vartheta_{1n}\\ 0 & 0 & \vartheta_{22} & \cdots & \vartheta_{2n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \vartheta_{nn}\end{bmatrix}\begin{bmatrix}p_0\\p_1\\.\\.\\.\\p_n\end{bmatrix}, \qquad \ldots(7)$$

where ϑ_{hh} (h=0, 1, …, n) are the power bases on which the (TPs)parameters are calculated, and the matrix is definitely invertible. Since, the derivative of equation (4) is:

$$y_n'(u) = \frac{d}{du}\sum_{a=0}^{n}\sigma\ (n,a)u^a = \sum_{a=1}^{n}\binom{n}{a}au^{a-1}, \binom{n}{a} = \frac{n!}{a!(n-a)!}, \qquad \ldots(8)$$

then, the derivative of equation (7) is:

$$N_n'(u) = [0\ 1\ 2u \ldots nu^{n-1}].\begin{bmatrix} \vartheta_{00} & \vartheta_{01} & \vartheta_{02} & \cdots & \vartheta_{0n} \\ 0 & \vartheta_{11} & \vartheta_{12} & \cdots & \vartheta_{1n} \\ 0 & 0 & \vartheta_{22} & \cdots & \vartheta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \vartheta_{nn} \end{bmatrix}.\begin{bmatrix} p_0 \\ p_1 \\ . \\ . \\ . \\ p_n \end{bmatrix} \qquad \ldots(9)$$

3. Solutions Generalized Abel Integral Equations

Let's say that using Tochard polynomials to find an approximate numerical solution to equation (1). By using equation (5), suppose that:

$$N(u) \cong N_n(u) = \sum_{a=0}^{n} p_a y_a\ (u), \qquad \ldots(10)$$

now, substituting the equation(10) into the equation(1) yields:

$$\sum_{a=0}^{n} p_a y_a\,(u) = m(u) + \int_{\rho}^{u} \frac{1}{(u-\mu)^{\tau}} \sum_{a=0}^{n} p_a y_a\,(\mu)\, d\mu, \qquad \ldots(11)$$

by using equation(6) then equation(11) becomes:

$$[y_0(u)y_1(u)\ldots y_n(u)].\begin{bmatrix} p_0 \\ p_1 \\ . \\ . \\ . \\ p_n \end{bmatrix} = m(u) + \int_{\rho}^{u} \frac{1}{(u-\mu)^{\tau}} [y_0(\mu)y_1(\mu)\ldots y_n(\mu)].\begin{bmatrix} p_0 \\ p_1 \\ . \\ . \\ . \\ p_n \end{bmatrix} d\mu, \qquad \ldots(12)$$

by using equation(7), then equation(12) becomes:

$$[1\, u\, u^2 \ldots u^n].\begin{bmatrix} \vartheta_{00} & \vartheta_{01} & \vartheta_{02} & \cdots & \vartheta_{0n} \\ 0 & \vartheta_{11} & \vartheta_{12} & \cdots & \vartheta_{1n} \\ 0 & 0 & \vartheta_{22} & \cdots & \vartheta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \vartheta_{nn} \end{bmatrix}.\begin{bmatrix} p_0 \\ p_1 \\ . \\ . \\ . \\ p_n \end{bmatrix}$$

$$= m(u)$$

$$+\int_{\rho}^{u} \frac{1}{(u-\mu)^{\tau}}[1\, \mu\, \mu^2 \ldots \mu^n].\begin{bmatrix} \vartheta_{00} & \vartheta_{01} & \vartheta_{02} & \cdots & \vartheta_{0n} \\ 0 & \vartheta_{11} & \vartheta_{12} & \cdots & \vartheta_{1n} \\ 0 & 0 & \vartheta_{22} & \cdots & \vartheta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \vartheta_{nn} \end{bmatrix}.\begin{bmatrix} p_0 \\ p_1 \\ . \\ . \\ . \\ p_n \end{bmatrix} d\mu, \qquad \ldots(13)$$

The unknown parameters are obtained by selecting u_σ $(\sigma = 0,1,\ldots,n)$ in the interval [0, 1] after calculating the integrations of equation (13). As a result, equation (13) becomes a (n+1) linear algebraic system of equations with (n+1) parameters (p_0, p_1, …, p_n)that are unknown can be solved using the "Gauss elimination method". Finally, by substituting these parameters into equation (5), the numerical solutions are obtained.

Note: Equations (2) and (3) can be solved using the same procedures.

4. Solution Accuracy

4.1 *For Generalized Abel Integral Equation*

The proposed method's accuracy is tested in this section. [15, 16, 17, 18, 19].Since equation (12) has the following form:

$$\sum_{a=0}^{n} p_a y_a(u) = m(u) + \int_{\rho}^{u} \frac{1}{(u-\mu)^{\tau}} \sum_{a=0}^{n} p_a y_a(\mu)\, d\mu, \qquad \ldots(14)$$

also equation (5) has the following formula:

$$N_n(u) = \sum_{a=0}^{n} p_a y_a(u), \quad 0 \le u \le 1$$

equation (13) was applied to find the unknown Touchard parameters $\{p_a\}_{a=0}^{n}$. Equation (10), on the other hand, gives us:

$$N(u) \cong N_n(u) = \sum_{a=0}^{n} p_a y_a(u), \qquad \ldots(15)$$

since this is the only polynomial solution of equation (13), and it's included into the equation (14). So, assume that $u = u_\gamma$ belong to the interval [0, 1], $\gamma = 0, 1, \ldots, n$. Now, the function of error can be expressed as follows:

$$Ef(u_\gamma) = \left| \sum_{a=0}^{n} p_a y_a(u_\gamma) - m(u_\gamma) - \int_{\rho}^{u_\gamma} \frac{1}{(u-\mu)^\tau} \sum_{a=0}^{n} p_a y_a(\mu)\, d\mu \right| \cong 0, \text{ then}$$

$Ef_n(u_\gamma) \leq \in$, for each u_γ in the interval [0, 1] and $\in > 0$.

As a result, at each point u_γ, the difference for error function $Ef(u_\gamma)$ will be smaller than any positive integer $\in > 0$. Then, using the relation, the error function $Ef(u)$ can be calculated:

$$Ef(u) = \sum_{a=0}^{n} p_a y_a(\theta) - m(u) - \int_{\rho}^{u} \frac{1}{(u-\mu)^\tau} \sum_{a=0}^{n} p_a y_a(\mu)\, d\mu,$$

thus, $Ef(u) \leq \in$.

Note: Equations (2) and (3) can be computed using the same procedure.

6. Illustrative Examples

This section includes some numerical examples to demonstrate the feasibility and efficacy of the suggested method for locating solutions utilizing matlabR2018b codes.

Example 1: Solve the (GAI) equation of the 1st type [1(page 246), 5, 20]

$$\frac{2}{3}\pi u^3 = \int_0^u \frac{1}{\sqrt{u^2 - \mu^2}} N(\mu)\, d\mu,$$

$N(u) = \pi u^3$ is the exact solution. By applying the proposed method in equation (13) for n=3 and selecting $u_0 = 0.1, u_1 = 0.2$ and $u_1 = 0.3$ from the interval [0, 1] and solving the system in "Gauss elimination" method, we obtained the following Touchard parameters, substituting these parameters in equation (5), we have the same given exact solution (approximate numerical solution) as follows:

$$N_3(u) = (-\pi) y_0(u) + (3\pi) y_1(u) + (-3\pi) y_2(u) + (\pi) y_3(u) = \pi u^3,$$

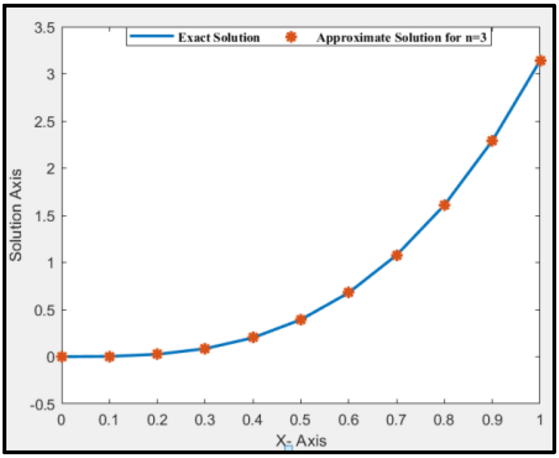

Figure 1

Example 1 (n = 3) is compared

In the reference [20], an exact solution for $n = 3$ was determined by using the Legendre polynomials approach. Moreover, [5] obtained 0.000211964 as the maximum error for n=5 by using Taylor collocation approach. As a result, our proposed method outperforms Taylor collocation method and it's identical in precision to Legendre polynomials approach. Figure1 presents the comparison with the exact solution for n=3.

Example 2: Solve the (GAI) equation of the 1st type [20, 21, 22, 23]

$$\frac{32768}{100947}u^{31/4} + \frac{262144}{908523}u^{27/4} + \frac{128}{231}u^{11/4} = \int_0^u \frac{u^2\mu^3 + \mu^4 + 1}{(u-\mu)^{1/4}} N(\mu)\, d\mu,$$

$$0 \le u \le 1$$

$N(u) = u^2$ is the exact solution. By applying the presented method in equation (13) for n=2 and selecting the points $u_0 = 0.1$, $u_1 = 0.2$ and $u_2 = 0.3$ from the interval [0, 1] and solving the system, we have the same given exact solution (approximate numerical solution) as follows:

$$N_2(u) = (1)y_0(u) + (-2)y_1(u) + (1)y_2(u) = u^2$$

which is the same exact solution of this example. [20], using Laguerre polynomials, an exact solution for n = 2 was obtained. Moreover, [21] using Jacobi collocation method, the exact solution for $n \ge 2$ was found. [22] obtained the highest number of order errors $3*10^{-3}, 2*10^{-2}$,

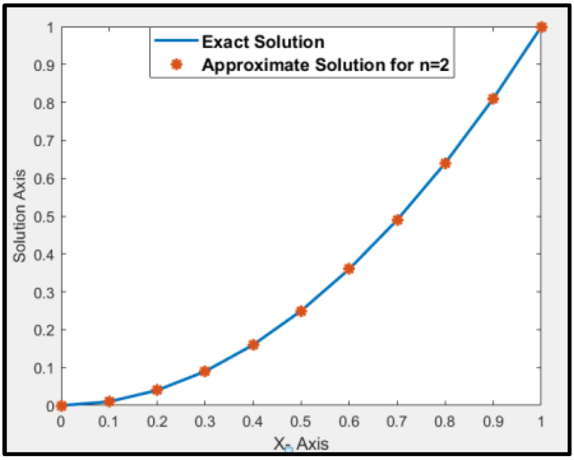

Figure 2

Example 2 (n = 2) is compared

$8*10^{-3}, 2*10^{-4}$, and $1*10^{-2}$ using the methods used in his research for h = 0.05. In addition to, [23] when h = 0.025, the mid-point, trapezoidal, Richardson extrapolation, and product integration procedures were used to produce the errors of orders $4*10^{-5}, -1*10^{-2}, -1*10^{-6}$, and $2*10^{-4}$ respectively. Our method is more efficient than the ones mentioned. Figure 2 presents the comparison with the exact solution for n=2.

Example 3: Solve the (GAI) equation of the 2nd type [1(page 250), 5]

$$N(u) = 1 - 2u - \frac{32}{21}u^{7/4} + \frac{4}{3}u^{3/4} - \int_0^u \frac{1}{(u-\mu)^{1/4}} N(\mu)\,d\mu, \quad 0 \le u \le 1$$

which has the exact solution $N(u) = 1 - 2u$. by applying the presented method in equation (13) with n = 3, selecting the points from the interval [0, 1] and solving the algebraic system, have the same given exact solution (approximate numerical solution) as shown in the following:

$$N_3(u) = (3)y_0(u) + (-2)y_1(u) = 1 - 2u,$$

also [5] obtained the exact solution by Taylor-collocation technique for n = 3. Figure3 presents the comparison with the exact solution for n=3.

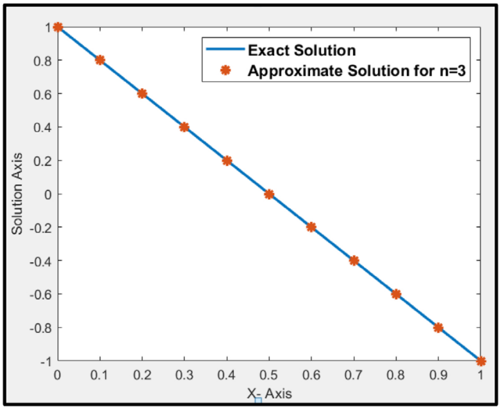

Figure 3

Example 3 (n = 3) is compared

Example 4: Solve the (GAID) equation of the 2nd type [10, 19]:

$$N'(u) = -N(u) - u + 0.2\int_0^u \frac{N'(\mu)+1}{\sqrt{(u-\mu)}}d\mu, \quad 0 < u \le 1, N(0) = 1,$$

where $N(u) = 1 - u$, is the exact solution.

By applying the presented method in equation (13) for n=2, 3 and 4, have the following Touchard parameters, substituting these parameters in equation (5), getting the same given exact solution (approximate numerical solution) as shown in the following:

$$N_2(u) = (2)y_0(u) + (-1)y_1(u) = 1 - u \ ,$$

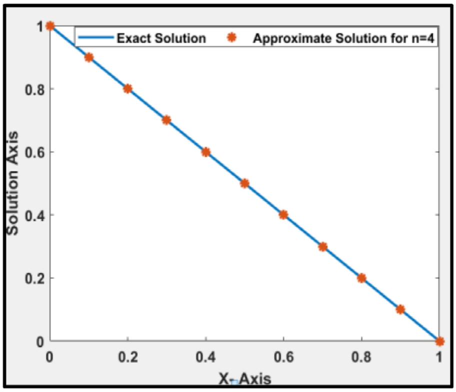

Figure 4

Example 4 (n = 4) is Compared

$$N_3(u) = (2)y_0(u) + (-1)y_1(u) = 1 - u \ ,$$
$$N_4(u) = (2)y_0(u) + (-1)y_1(u) = 1 - u$$

The reference [19], for n = 2, 3, and 4, an exact solution was obtained by applied Laguerre and Touchard polynomials. Besides, [10] used the Chebyshev polynomials for n = 8, 12, and 16 to find the maximum absolute error of order 10^{-17}. Figure 4 presents the comparison with the exact solution for n= 4

7. Conclusions

In this manuscript, the technique based on (TPs) has been used to solve (GAI) equations of the first, second type and (GAID) equation. A different set of points belonging to the given interval [0, 1] and many different degrees of Touchard polynomials were used. Four examples were solved, two of which were (GAI) equations of the first type, one of the second type and the fourth was (GAID) equation using the Matlab R2018b. When all of the examples were solved, exact solutions were obtained, and the results were compared to many of the approaches described in the literature. All of the results in the proposed technique were compared to the given exact solutions of examples via graphs, demonstrating the method's effectiveness and accuracy.

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